Equivariant Dynamical Systems in Infinite Dimensions

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Trends in Bifurcation Analysis
Methods and Applications
Milano, June 3-5, 2009

Universität Bielefeld

SFB 701

[1] Based on joint work with
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Overview

- PDEs as dynamical systems in infinite dimensions
- Dynamics of patterns on unbounded domains
- Traveling and rotating waves
- Equivariant evolution equations
- Relative equilibria and relative periodic orbits
- Stability with asymptotic phase
- The freezing method
- Continuation and bifurcation
- Summary and perspectives
- References

PDEs as dynamical systems in infinite dimensions

A quotation from the classical text by `D.Henry (1981)`:

'In 1971, I read the beautiful paper of Kato and Fujita (1964) on the Navier-Stokes equation and was delighted to find that, properly viewed, it looked like an ordinary differential equation, and the analysis proceeded in ways familiar for ODEs. This is perhaps no surprise to people in partial differential equations, but my training was in ...'

In a sense this applies equally to the numerical analysis of time-dependent PDEs with extra difficulties arising from

- spatial discretization (high-dimensional sparse ODEs),
- unbounded spectra and spurious eigenvalues (stiff ODEs),
- spatio-temporal interaction (mesh adaptation in time and space).

The model problem: Chafee Infante

$$u_t = u_{xx} + f(u,\lambda), \quad f(u,\lambda) = \lambda u(1-u)(1+u)$$
$$u(x,0) = u_0(x)\ (-1 \le x \le 1), \quad u(0,t) = u(1,t) = 0\ (t \ge 0)$$

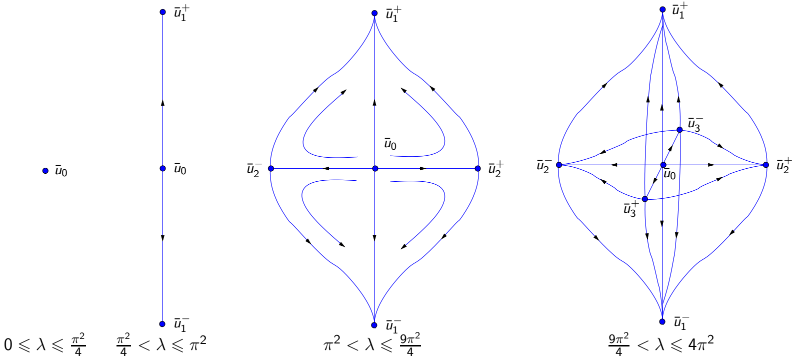


Figure: Equilibria and phase diagram with increasing λ

Evolution equation

$$\boxed{\begin{array}{c} u_t = F(u,\lambda), \quad u(0) = u_0, \qquad F : D(F) \subset X \to X, \\ X \text{ Banach space}, D(F) \text{ dense subspace} \\ u(t) = \varphi^t(u_0), t \geq 0 \quad \text{solution semi-flow} \end{array}} \qquad \text{(EVOL)}$$

Here

$$\boxed{F(u,\lambda) = u_{xx} + f(u,\lambda), \quad u \in D(F) = H^2(-1,1) \cap H_0^1(-1,1)}$$

Weak formulation for $u(t) \in H_0^1(-1,1)$

$$\boxed{\frac{d}{dt}(u,v) = -(u_x, v_x) + (f(u,\lambda), v) \quad \text{for all } v \in H_0^1(-1,1).}$$

Neumann boundary conditions: replace $H_0^1(-1,1)$ by $H^1(-1,1)$, note the change in bifurcation diagram.

Finite element discretization

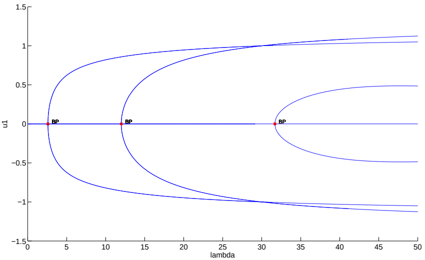
Finite dimensional space of test functions

$$V_{\Delta x} \subset H^1_0(-1,1), \quad \text{e.g. piecewise linear on a grid of width } \Delta x$$

Discretized solution flow

$$\boxed{u_{\Delta x}(t) = \Phi^t_{\Delta x}(u_0) \in V_{\Delta x}}$$

$$\boxed{\begin{aligned} &\frac{d}{dt}(u,v) = -(u_x, v_x) + (f(u,\lambda), v) \quad \text{for all } v \in V_{\Delta x} \\ &(u(0), v) = (u_0, v) \quad \text{for all } v \in V_{\Delta x} \end{aligned}}$$



Bifurcation diagram of FE-discretization first coefficient from $u = \sum_{j=1}^N u_j v_j$ $V_{\Delta x} = \mathrm{span}\{v_j\}_{j=1}^N$.

Dynamics of patterns on unbounded domains

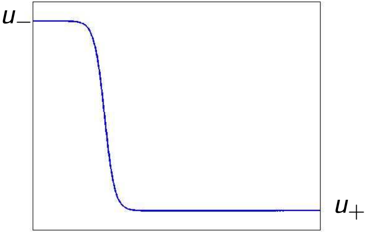
Parabolic system

$$\boxed{u_t = Au_{xx} + f(u, u_x), \qquad x \in \mathbb{R},\ u(x,t) \in \mathbb{R}^m} \qquad \text{(PDE)}$$

where $A \in \mathbb{R}^{m,m}$ pos. def., $f : \mathbb{R}^{2m} \to \mathbb{R}^m$ smooth, $f(u_\pm, 0) = 0$

Traveling waves: $u(x,t) = \bar{v}(x - \bar{\mu}t)$, profile $\bar{v}$, velocity $\bar{\mu}$

traveling front: $\lim_{x\to\pm\infty} \bar{v}(x) = u_\pm$

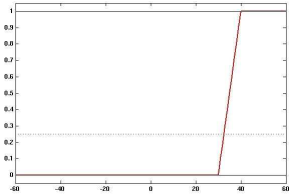


Nagumo wave

$$u_t = u_{xx} + u(1-u)(u-\alpha),$$
$$\bar{v}(x) = \frac{1}{1+e^{-x/\sqrt{2}}}, \quad \bar{\mu} = -\sqrt{2}(\frac{1}{2}-\alpha)$$

profile velocity

Cauchy problem: $u(x,0) = u_0(x), |x| \le 60, \alpha = \frac{1}{4}$, Neumann b.c.



Note: $f(w,\lambda) = \lambda(w-u_1)(u_2-w)(w-u_3), u_1 < u_2 < u_3$
transforms into the Nagumo equation via

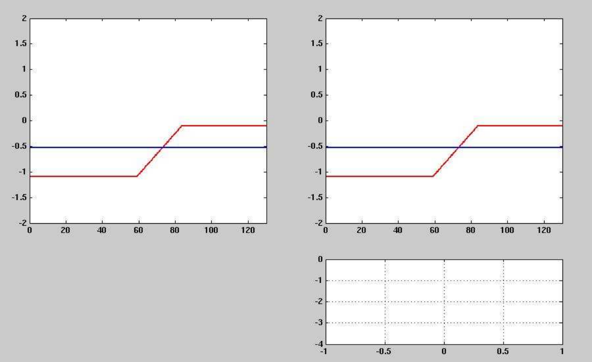
$$w(x,t) = u_1 + u(sx, s^2 t)(u_3 - u_1),\ \alpha = \frac{u_2-u_1}{u_3-u_1},\ s^2 = \lambda(u_3-u_1)^2.$$

Traveling and rotating waves

FitzHugh-Nagumo wave

$$V_t = \Delta V + V - \frac{1}{3}V^3 - R,$$
$$R_t = \phi(V + a - bR)$$

$J = [0, 130]$, $\Delta x = 0.5$, $\Delta t = 0.01$,$a = 0.7$,$b = 0.8$, $\phi = 0.08$.



traveling vs. comoving observer

Quintic Ginzburg Landau equation 1d

$$u_t = \alpha u_{xx} + \delta u + \beta |u|^2 u + \gamma |u|^4 u,$$
$$u(x,t) \in \mathbb{C}$$

$\alpha = 1,\ \delta = -0.1,\ \beta = 3 + i,\ \gamma = -2.75 + i,\ G = \mathbb{R} \times S^1$

Rotating and traveling waves $u(x,t) = e^{-i\mu_{\mathrm{rot}} t}\bar{v}(x - \mu_{\mathrm{trav}} t)$

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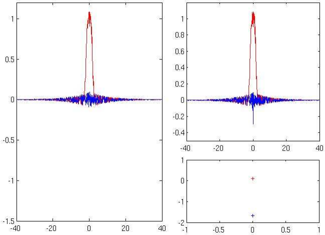
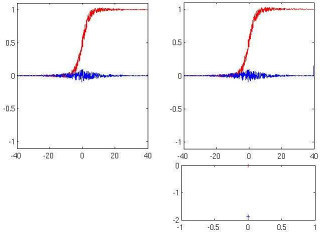
Numerical solutions $h = 0.1,\ J = [-40, 40],$ Neumann bc.



rotating pulse



rotating and traveling front

Reaction diffusion systems in $\mathbb{R}^2$

$$\boxed{\begin{aligned} u_t &= \Delta u + f(u), \quad t \geq 0 \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}^2 \end{aligned}}$$

Patterns rotating about a center $z \in \mathbb{R}^2$

$$\boxed{\begin{aligned} u(x,t) &= \bar{v}(R_{-\bar{\mu}t}(x-z)+z) \\ R_\phi &= \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \end{aligned}}$$

$\bar{v}$ profile, $\bar{\mu}$ angular velocity.
A particular spot $\bar{v}(\xi)$ of the profile traces the curve $x(t)$

$$\xi = R_{-\bar{\mu}t}(x(t)-z)+z \Longleftrightarrow x(t) = R_{\bar{\mu}t}(\xi - z) + z.$$

Example: Quintic Ginzburg Landau equation in 2D

$$u_t = \alpha \Delta u + \delta u + \beta |u|^2 u + \gamma |u|^4 u, \quad (x,y) \in \mathbb{R}^2,\ u(x,y,t) \in \mathbb{C}$$

$\alpha = 0.5(1+i),\ \delta = -0.5,\ \beta = 2.5 + i,\ \gamma = -1 - 0.1i,$
see `Crasovan, Malomed, Michalache 2001` (spinning solitons).

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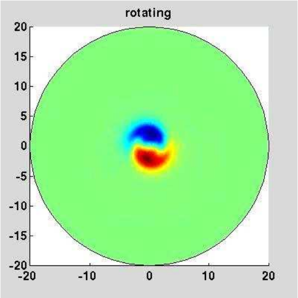
Finite element package (Comsol Multiphysics), Neumann b.c.



Equivariant evolution equations

$$\boxed{\begin{array}{c} u_t = F(u), \quad u(0) = u_0, \qquad F : Y \subset X \to X, \\ X \text{ Banach space, } Y \text{ dense subspace} \end{array}} \qquad \text{(EVOL)}$$

G is a (noncompact) Lie group acting on X via

$$\boxed{a : G \times X \to X,\ (\gamma, v) \mapsto a(\gamma)v}$$

Assumptions: $a(\gamma) \in \mathrm{GL}(X)$ and

$$a(\mathbb{1})v = v, \quad a(\gamma_1 \circ \gamma_2)v = a(\gamma_1)a(\gamma_2)v$$

$a(\cdot)v : \ \gamma \mapsto a(\gamma)v$ continuous $\forall v \in X$,

differentiable $\forall v \in Y$ with derivative $d[a(\gamma)v]$

Equivariance

$$\boxed{\begin{array}{c} a(\gamma)(Y) \subset Y \quad \forall \gamma \in G \\ F(a(\gamma)u) = a(\gamma)F(u) \quad \forall u \in Y, \gamma \in G \end{array}}$$

Examples: real valued case

$$u_t = A\Delta u + f(u), \quad u(x,t) \in \mathbb{R}^m, x \in \mathbb{R}^d, t \geq 0$$
$$A \in \mathbb{R}^{m,m}, \quad f : \mathbb{R}^m \to \mathbb{R}^m$$

Special Euclidean Group: $G = SE(d) = SO(d) \ltimes \mathbb{R}^d$
group operation: $(R_1, \tau_1) \circ (R_2, \tau_2) = (R_1 R_2, R_1 \tau_2 + \tau_1)$,
group action: $[a(\gamma)u](x) = u(\gamma x),\ x \in \mathbb{R}^d$.
function spaces:

- $X = \mathcal{L}^2(\mathbb{R}^d), Y = \mathcal{H}^2(\mathbb{R}^d)$ (strongly localized patterns)
- $Y = H^2_{Eucl} = \{u \in \mathcal{H}^2 : x_i D_j u - x_j D_i u \in \mathcal{L}^2 \ \forall i, j = 1, \ldots, d\}$.
- $X = C_{unif}, Y = C^2_{unif}$ or $Y = C^2_{unif,Eucl}$

Examples: complex valued case

$$u_t = (A\Delta + g(|u|))u, \quad u(x,t) \in \mathbb{C}^m, x \in \mathbb{R}^d, t \geq 0$$
$$A \in \mathbb{C}^{m,m}, \quad g : \mathbb{R} \to \mathbb{C}$$

May be rewritten as a real system of double dimension

Group: $G = S^1 \times SE(d)$
group action: $[a(\theta,\gamma)u](x) = e^{i\theta}u(\gamma x), \; x \in \mathbb{R}^d$.
function spaces: $X = \mathcal{L}^2$, $Y = \mathcal{H}^2$, as above.

General Theory:
`Chossat, Lauterbach 2000`, `Golubitsky, Stewart 2002`, `Field 2007`.

Relative equilibria and relative periodic orbits

Definition: A solution $u(t) = a(\gamma(t))\bar{v}, \bar{v} \in Y, \gamma \in C^1(\mathbb{R}_+, G)$ of $u_t = F(u)$ is called a relative equilibrium.
Likewise, a solution $u(t) = a(\gamma(t))\bar{v}(t)$ is called a relative periodic orbit if $\bar{v}(t)$ has a nontrivial period $T > 0$.

Examples

- Traveling wave : $u(x,t) = \bar{v}(x - \bar{\mu}t)$, $\gamma(t) = \bar{\mu}t$, $G = \mathbb{R}$,
- Rotating pattern: $u(x,t) = \bar{v}(R_{-\bar{\mu}t}(x - z) + z)$, $\gamma(t) = R_{-\bar{\mu}t}(x - z) + z$, $G = SE(2)$

Characterization of relative equilibria

Recall

- the unit element $\mathbb{1} \in G$,
- the Lie algebra $\mathcal{A} = T_{\mathbb{1}} G$ associated with G,
- the derivative $dL_\gamma(\mathbb{1}) : \mathcal{A} \to T_\gamma G$ of $L_\gamma g = \gamma \circ g$,
- the derivative of group action $d[a(\gamma)v] : T_\gamma G \mapsto X$

Theorem

The function $a(\gamma(t))\bar{v}, t \geq 0$ is a relative equilibrium iff there exists some $\bar{\mu} \in \mathcal{A}$ such that $\bar{v}$ is a steady state of

$$\boxed{v_t = F(v) - d[a(\mathbb{1})v]\bar{\mu}} \tag{PDEstat}$$

and

$$\boxed{\gamma_t = dL_\gamma(\mathbb{1})\bar{\mu}} \tag{RE}$$

Remarks:

- If $d[a(\mathbb{1})\bar{v}] : \mathcal{A} \mapsto X$ is one-to-one then $\bar{\mu}$ is unique.
- Eqn. (RE) has solution $\gamma(t) = \exp(t\bar{\mu})\gamma(0)$.

Stability with asymptotic phase

Definition

A relative equilibrium $a(\gamma(t))\bar{v}$ is called stable with asymptotic phase if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for all solutions of $u_t = F(u)$ with $||u(0) - \bar{v}|| \leq \delta$ there exists $\gamma_0(t) \in G$ satisfying

$$\|u(t) - a(\gamma_0(t))\bar{v}\| \begin{cases} \leq \epsilon, & \forall t \geq 0 \\ \to 0 & \text{for } t \to \infty. \end{cases}$$

Stability of a traveling wave $u(x,t) = \bar{v}(x - \bar{\mu}t)$ for

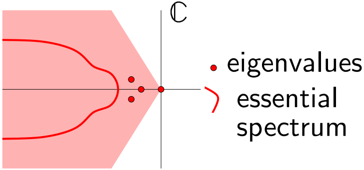
$$\boxed{u_t = Au_{xx} + f(u, u_x), \qquad x \in \mathbb{R},\ u(x,t) \in \mathbb{R}^m} \qquad \text{(PDE)}$$

depends on spectral properties of the linearized operator

$$\Lambda v = Av'' + (\bar{\mu}I + D_2 f(\bar{v}, \bar{v}'))v' + D_1 f(\bar{v}, \bar{v}')v \quad \text{in } \mathcal{L}_2$$

Stability of $(\bar{v}, \bar{\mu})$ is determined by spectral properties of

$$\Lambda v = Av'' + (\bar{\mu} I + D_2 f(\bar{v}, \bar{v}'))v' + D_1 f(\bar{v}, \bar{v}')v \quad \text{in } \mathcal{L}_2$$



Theorem (Stability of traveling waves)

Spectral behavior of Λ *(*0 *is a simple eigenvalue and all other eigenvalues as well as essential spectrum satisfy* $\mathrm{Re} \leq -\beta < 0$*) and growth conditions and smoothness of* f *imply that* $(\bar{v}, \bar{\mu})$ *is stable with asymptotic phase with respect to* $||\cdot||_{\mathcal{H}^1}$*.*

Remark: See Henry 1981, Bates,Jones 1989, Volpert et al. 1994 , Sandstede 2002,...

Stability of rotating localized patterns

(WJB,Lorenz 2008)

$$u_t = A\Delta u + f(u), \quad u(x,0) = u_0(x) \quad x \in \mathbb{R}^2,\ t \geq 0, \quad u(x,t) \in \mathbb{R}^m \tag{RD}$$

Rigidly rotating localized wave $\bar{v}$

$$u(x,t) = \bar{v}(R_{-\bar{\mu}t}x), \bar{\mu} > 0, R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{LP}$$

- $f \in C^4$,
- $\sup_{|x|\geq r, |\alpha|\leq 2} |D^\alpha \bar{v}(x)| \to 0 \quad \text{as} \quad r \to \infty$,
- $f'(0) \leq -2\beta I < 0$,
- the eigenvalues $\pm i\bar{\mu}$ with ev $D_1\bar{v} \pm D_2\bar{v}$ and 0 with ev $D_\phi\bar{v}$ are algebraically simple for the linearized operator $\mathcal{L} = A\Delta + \bar{\mu}D_\phi + f'(\bar{v})$ in $H^2_{Eucl} = \{u \in H^2 : D_\phi u \in L^2\}$,
- $\mathcal{L}$ has no further eigenvalues $s \in \mathbb{C}$ with $\mathrm{Re}\,(s) \geq -2\beta$.

Nonlinear stability for rotating localized $2D$-patterns

Theorem

Under the assumptions above there exists an $\varepsilon > 0$ such that for any solution of (RD) *satisfying* $||u_0 - \bar{v}||_{\mathcal{H}^2} \leq \varepsilon$ *there exist a* C^1*-function* $\gamma(t) = (\theta(t), \tau(t)) \in SE(2)$ *and some* $(\theta_\infty, \tau_\infty) \in SE(2)$ *such that*

$$\boxed{\begin{aligned} ||u(\cdot,t) - a(\gamma(t))\bar{v}||_{\mathcal{H}^2} &\leq C \exp(-\beta t)||u_0 - \bar{v}||_{\mathcal{H}^2} \\ |\theta(t) + \bar{\mu} t - \theta_\infty| + |\tau(t) - \tau_\infty| &\leq C \exp(-\beta t)||u_0 - \bar{v}||_{\mathcal{H}^2} \end{aligned}}$$

Remark: The system (RD) is equivariant with respect to the action of $SE(2) = S^1 \ltimes \mathbb{R}^2 \ni (\theta, \tau)$:

$$\boxed{a(\theta,\tau)u(x) = u(R_{-\theta}(x - \tau)), \quad x \in \mathbb{R}^2}$$

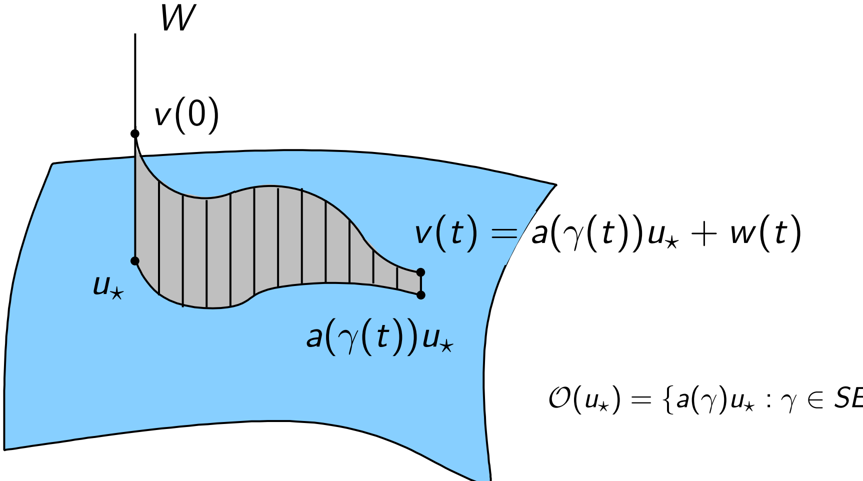
See `Sandstede,Scheel,Wulff 1999` for a center manifold reduction.

Ingredients from the proof

Transform into rotating coordinates

$$\boxed{v_t = A\Delta v + \bar{\mu} D_\phi v + f(v),}$$

Decompose the solution $v(x,t) = \bar{v}(R_{-\theta(t)}(x - \tau(t)) + w(x,t)$



Eigenvalues and essential spectrum of linearization

Transform to rotating coordinates $u(x,t) = v(R_{-\bar{\mu}t}x, t)$

$$\boxed{v_t = A\Delta v + \bar{\mu} D_\phi v + f(v), \quad \text{with} \quad D_\phi v = -x_2 D_1 v + x_1 D_2 v.}$$

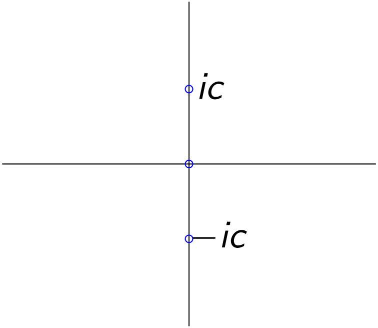
Linearize at the steady state $v = \bar{v}$

$$\boxed{\mathcal{L}u = A\Delta u + \bar{\mu} D_\phi u + B(x)u, \quad B(x) = f'(\bar{v}(x)).}$$

Apply D_1, D_2, D_ϕ to $A\Delta\bar{v} + \bar{\mu} D_\phi \bar{v} + f(\bar{v}) = 0$

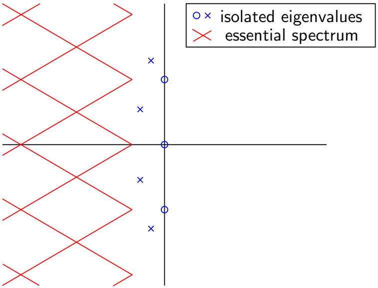
$$\boxed{0 = \mathcal{L}D_\phi \bar{v} = \mathcal{L}(D_1\bar{v}) + \bar{\mu} D_2\bar{v} = \mathcal{L}(D_2\bar{v}) - \bar{\mu} D_1\bar{v},}$$

in particular, $\mathcal{L}(D_1\bar{v} \pm iD_2\bar{v}) = \pm i\bar{\mu}(D_1\bar{v} \pm iD_2\bar{v})$



critical eigenvalues: $0, \pm ic$

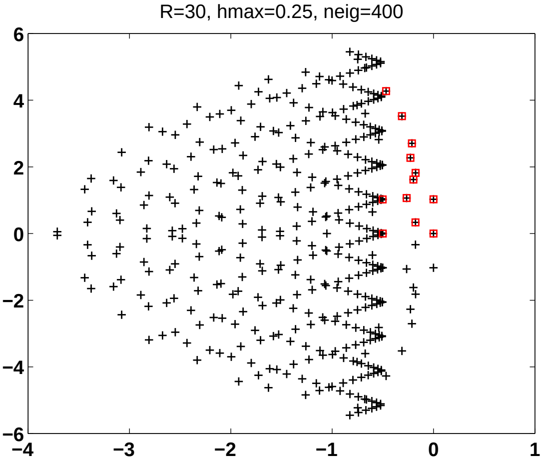
Generic picture of full spectrum



Essential spectrum, critical eigenvalues, and further isolated eigenvalues

Semigroup $e^{t\mathcal{L}}$ is continuous but not analytic !

Part of numerical spectrum: 400 ev, system size $\approx 10^4$



8 additional pairs of isolated eigenvalues.

Essential spectrum

Consider at $r = \infty$

$$\mathcal{L} = A\left(D_r^2 + \frac{1}{r}D_r + \frac{1}{r^2}D_\phi^2\right) + \bar{\mu}D_\phi + f'(u_*(r,\phi))$$

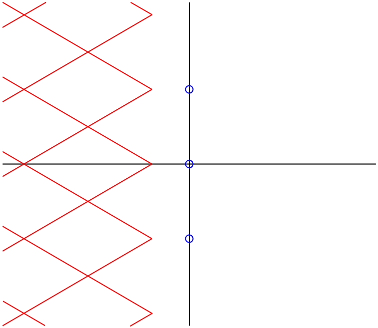
With $u(r,\phi) = e^{in\phi}e^{i\kappa r}u_\infty$ obtain $s \in \sigma_{\text{ess}}$, if s satisfies for some $\kappa \in \mathbb{R}$ and $n \in \mathbb{Z}$

$$\det(-\kappa^2 A + in\bar{\mu} + f'(0) - s) = 0 \quad \text{dispersion relation}$$

Quintic Ginzburg Landau: ∞ many copies of two half lines

$$s = -\kappa^2\alpha + in\bar{\mu} + \delta, \quad s = -\kappa^2\bar{\alpha} + in\bar{\mu} + \bar{\delta}, \quad \kappa \in \mathbb{R}, \quad n \in \mathbb{Z}.$$

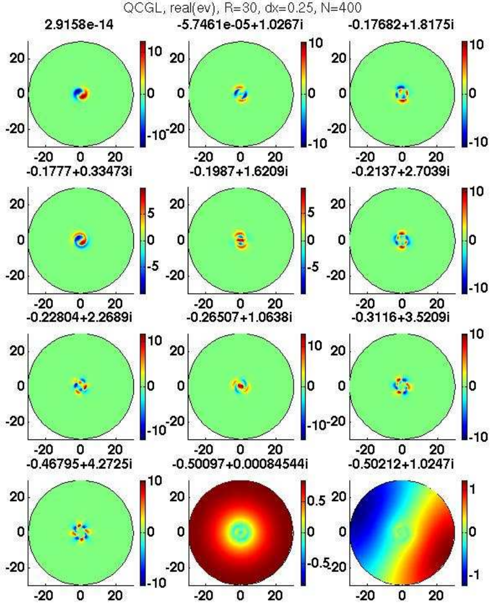
Quintic Ginzburg Landau: $\alpha = \frac{1+i}{2}, \delta = -\frac{1}{2} < 0$



essential spectrum: $s = inc + \delta - \kappa^2(\alpha_1 \pm i\alpha_2), \kappa \in \mathbb{R}, n \in \mathbb{Z}$

Real parts of eigenfunctions

2 critical and 8 extra isolated eigenvalues, 2 'non-eigenvalues'



The freezing method

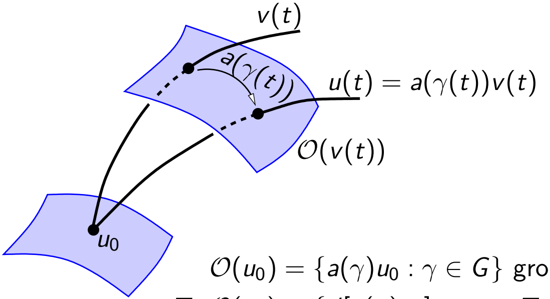
Introduce $\gamma(t) \in G$, $v(t) \in Y$ and write

$$\boxed{u(t) = a(\gamma(t))v(t)}$$

Add a phase condition on v

$$\boxed{\psi(v) = 0, \quad \text{where } \psi : Y \mapsto \mathcal{A}^*}$$

$\mathcal{A}^* = \mathcal{L}[\mathcal{A}, \mathbb{R}]$ dual of $\mathcal{A} = T_{\mathbb{1}} G$ (Lie algebra)



$\mathcal{O}(u_0) = \{a(\gamma)u_0 : \gamma \in G\}$ group orbit

$T_{u_0}\mathcal{O}(u_0) = \{d[a(\gamma)u_0]\mu : \mu \in T_\gamma G\}$ tangent space

The 'frozen' PDAE

Insert $u(t) = a(\gamma(t))v(t)$ into $u_t = F(u)$ (EVOL),
introduce $\mu(t) = \mu \in \mathcal{A}$ via $\gamma_t = dL_\gamma(\mathbb{1})\mu$
and then solve for $\gamma(t) \in G, \mu(t) \in \mathcal{A}, v(t) \in Y$
the differential algebraic evolution equation

$$\boxed{\begin{array}{ll} v_t = F(v) - d[a(\mathbb{1})v]\mu, & v(0) = u_0 \\ \gamma_t = dL_\gamma(\mathbb{1})\mu, & \gamma(0) = \mathbb{1} \\ 0 = \psi(v, \mu) & \text{phase conditions} \end{array}} \qquad \text{(DAEVOL)}$$

- The term $d[a(\mathbb{1})v]\mu$ in (DAEVOL)is obtained by applying $\frac{d}{dg}$ to $a(\gamma)a(g)v = a(L_\gamma g)v$ at $g = \mathbb{1}$

$$\boxed{a(\gamma)d[a(\mathbb{1})v]\mu = d[a(\gamma)v]dL_\gamma(\mathbb{1})\mu \quad \text{for all } \mu \in \mathcal{A}.}$$

- related approach: Rowley, Kevrekidis, Marsden and Lust 2003
- A relative equilibrium $\bar{v}, \bar{\mu}$ appears as equilibrium of (DAEVOL) when suitably normalized.

Realization for PDEs:

Choose basis $\{e^1, \dots, e^s\}$ in $\mathcal{A}$, $s = \dim G$, define

$$\mu = \sum_{i=1}^{s} \mu_i e^i, \quad -d[a(\mathbb{1})v]\mu = S(v)\mu = \sum_{i=1}^{s} S^i(v)\mu_i, \; S^i(v) = -\, da(\mathbb{1})ve^i$$

Phase conditions

$$\psi_{\rm fix}(v) = (\langle S^i(\hat{v}), v - \hat{v}\rangle)_{i=1}^{s} \in \mathbb{R}^s, \qquad \text{min. } \|v - a(\gamma)\hat{v}\|_{\mathcal{L}_2}$$

$$\psi_{\rm orth}(v,\mu) = (\langle S^i(v), F(v) - S(v)\mu\rangle)_{i=1}^{s} \in \mathbb{R}^s \qquad \text{min. } \|v_t\|_{\mathcal{L}_2}$$

PDE realization

$$\boxed{\begin{aligned} v_t &= A\Delta v + f(v, \nabla v) + S(v)\mu \\ 0 &= \psi(v,\mu) \end{aligned}}$$

Examples

$$G = \mathbb{R}, \; [a(\gamma)u](x) = u(x-\gamma), \; S(v)\mu = \mu v_x$$

$$G = \mathbb{R} \times S^1, \; [a(\gamma)u](x) = e^{-i\gamma_2} u(x - \gamma_1), \; S(v)\mu = \mu_2 i v + \mu_1 v_x$$

Freezing the quintic Ginzburg Landau equation, 2D

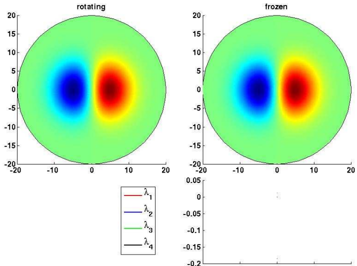
$$u_t = \alpha \Delta u + \delta u + \beta |u|^2 u + \gamma |u|^4 u, \quad (x,y) \in \mathbb{R}^2,\ u(x,y,t) \in \mathbb{C}$$

$\alpha = 0.5(1+i),\ \delta = -0.5,\ \beta = 2.5 + i,\ \gamma = -1 - 0.1i,$
$a(\gamma)v(\xi) = e^{i\theta} v(R_{-\phi}(\xi - \tau))$ for $\gamma = (\phi, \tau, \theta) \in G = SE(2) \times S^1,$

Freezing the quintic Ginzburg Landau equation, 2D

$$
\begin{aligned}
u_t &= \alpha \Delta u + \delta u + \beta |u|^2 u + \gamma |u|^4 u \\
&\quad + \mu_1 (y u_x - x u_y) + \mu_2 u_x + \mu_3 u_y + \mu_4 i u \\
0 &= \langle y u_{0,x} - x u_{0,y}, u - u_0 \rangle_{\mathcal{L}_2}, \quad 0 = \langle i u_0, u - u_0 \rangle_{\mathcal{L}_2} \\
0 &= \langle u_{0,x}, u - u_0 \rangle_{\mathcal{L}_2}, \quad 0 = \langle u_{0,y}, u - u_0 \rangle_{\mathcal{L}_2}
\end{aligned}
$$

$\alpha = 0.5(1+i)$, $\delta = -0.5$, $\beta = 2.5 + i$, $\gamma = -1 - 0.1i$,
$a(\gamma)v(\xi) = e^{i\theta} v(R_{-\phi}(\xi - \tau))$ for $\gamma = (\phi, \tau, \theta) \in G = SE(2) \times S^1$,
Computation with finite element package (Comsol Multiphysics), Neumann b.c.



Scroll waves in $\mathbb{R}^3$: CGL or $\lambda - \omega$-system

$$u_t = \Delta u + (1 - |u|^2 - i|u|^2)u, \quad x \in \mathbb{R}^3,\ u(x,t) \in \mathbb{C}$$

Action of Euclidean group

$$G = SE(3) = SO(3) \ltimes \mathbb{R}^3,\ \gamma = (R, \tau)$$
$$[a(\gamma)v](x) = v(R^{-1}(x - \tau))$$

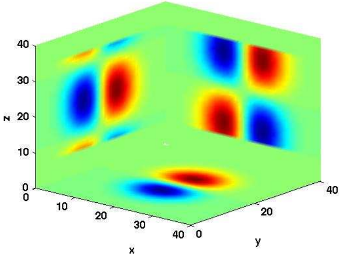
group operation $\gamma \circ \tilde{\gamma} = (R\tilde{R}, \tau + R\tilde{\tau})$

$$v_t = \Delta v + (1 - |v|^2 - i|v|^2)v + \mu_4 v_{x_1} + \mu_5 v_{x_2} + \mu_6 v_{x_3}$$
$$+\mu_1(v_{x_2}x_3 - v_{x_3}x_2) + \mu_2(v_{x_3}x_1 - v_{x_1}x_3) + \mu_3(v_{x_1}x_2 - v_{x_2}x_1)$$

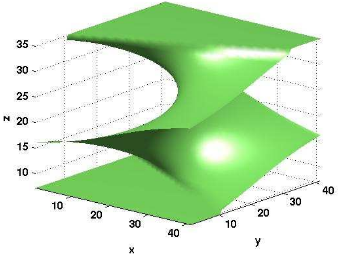
corresponding phase conditions

- Numerical solution with adaptation of `ezscroll` (Barkley '97)
- data: $L_{x_i} = 40$, $\Delta x_i = 1$, $\Delta t = \frac{3}{8}10^{-3}$, 19-point Laplacian
- boundary conditions: x, y - Neumann, z - periodic
- initial function:
  $u_0(r, \varphi, z) = \exp(\frac{iz}{2\pi})\frac{r}{40}(\cos(\varphi) + i\sin(\varphi))$

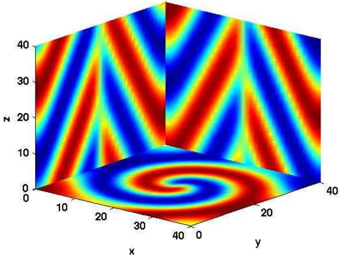
Scroll wave in 3D



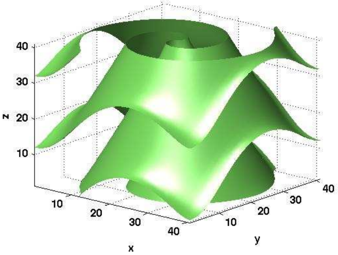
initial cond., x,y,x-slices through origin



isosurface, Re(u)=0



solution at $t = 300$, x,y,x-slices through origin



isosurface, Re(u)=0

Asymptotic stability and freezing

Goal: Stability of relative equilibrium with asymptotic phase turns into Liapunov stability of the pair $(\bar{v}, \bar{\mu})$.

(Meta)Theorem: Let $a(\gamma(t))\bar{v}$ be a relative equilibrium such that $\gamma(t) = \exp(\bar{\mu}t)$ and $\psi(\bar{v}, \bar{\mu}) = 0$. Assume nondegeneracy conditions for $\bar{v}, \bar{\mu}, \psi$.
Then stability with asymptotic phase holds if and only if $(\bar{v}, \bar{\mu}) \in Y \times \mathcal{A}$ is an asymptotically stable equilibrium of the system

$$\boxed{\begin{aligned} v_t &= F(v) - d[a(\mathbb{1})v]\mu, & v(0) &= u_0 \\ \gamma_t &= dL_\gamma(\mathbb{1})\mu, & \gamma(0) &= \mathbb{1} \\ 0 &= \psi(v, \mu) \end{aligned}} \qquad \text{(DAEVOL)}$$

in the Liapunov sense for all consistent initial data u_0.

Specific result

$$\boxed{\begin{aligned} v_t &= Av_{xx} + f(v, v_x) + \mu v_x, \quad v(\cdot, 0) = u_0 \\ 0 &= \psi_{\text{fix}}(v) = \langle \hat{v}_x, v(\cdot, t) - \hat{v} \rangle \quad \text{phase condition} \\ \gamma_t &= \mu(t), \quad \gamma(0) = 0 \end{aligned}} \qquad \text{(PDAE)}$$

Theorem (Stability for (PDAE))

Let the same spectral conditions on Λ, growth conditions on f as in the stability theorem be satisfied and assume $\langle \hat{v}', \bar{v}' \rangle \neq 0$. Then the traveling wave $(\bar{v}, \bar{\mu})$ is an asymptotically stable solution of (PDAE) *for the phase condition ψ_{fix}.*

Extensions:

- fully discrete scheme: finite interval, two-point boundary conditions, finite differences `Thümmler 2005,2008`
- phase condition ψ_{orth}, `WJB,Thümmler 2007`.

Continuation and bifurcation

Perform continuation and branch switching with respect to solutions $(v, \mu, \lambda) \in Y \times \mathcal{A} \times \mathbb{R}$ of

$$\boxed{\begin{aligned} 0 &= F(v,\lambda) - d[a(\mathbb{1})v]\mu, \\ 0 &= \psi(v,\mu), \end{aligned}}$$

cf. `Champneys,Sandstede 2007`.

- Turning and branching points: loss of stability for traveling waves
- Hopf points: Transition to modulated traveling waves resp. meandering spirals

Relative periodic orbits $((v(t), \mu(t))_{t\in[0,1]}, \lambda, T)$ from the BVP

$$\boxed{\begin{aligned} v_t &= T\left[F(v,\lambda) - d[a(\mathbb{1})v]\mu\right], v(0) = v(1), \mu(0) = \mu(1) \\ 0 &= \psi(v)\rho = \int_0^1 \langle d[a(\mathbb{1})\hat{v}]\rho, v - \hat{v}\rangle dt \quad \forall \rho \in \mathcal{A} \\ 0 &= \Psi(v) = \int_0^1 \langle \hat{v}_t, v - \hat{v}\rangle dt \end{aligned}}$$

Summary

- Spatio-temporal patterns can exist for all times on unbounded domains (relative equilibria of equivariant evolution equations),
- Relative equilibria are stable with asymptotic phase, if point and essential spectra (except for the trivial eigenvalues on the imaginary axis) lie strictly in the left half plane, proved for traveling waves in 1D and localized rotating waves in 2D,
- Freezing allows to adaptively compute moving coordinate systems for equivariant PDEs in 1D-3D,
- Leads to (P)DAEs of index 1 or 2 with artificial convective terms (can create problems in discretizations),
- Stability with asymptotic phase turns into Lyapunov stability,
- For parabolic systems in 1D the effects of the transformation

  $\text{PDE} \xrightarrow{\text{freezing}} \text{PDAE} \xrightarrow{\text{discretization}} \text{DAE}$ have been analyzed.

Perspectives

- freezing of relative equilibria for different types of equations
  - viscous conservation laws,
  - hyperbolic equations (balance laws), `Rottmann-Matthes 2009`
  - parabolic PDEs with nonlocal diffusion term,
  - SPDEs,
- More general equivariance $b(\gamma)F(u) = F(a(\gamma)u)$ needs rescaling of time, cf. `Rowley et al.2003`,
- Stability with asymptotic phase for nonlocalized (spiral) waves in dimension ≥ 2,
- Stability proof for the PDAE formulation in dimension ≥ 2,
- Systems with relative periodic orbits and their direct computation (`Wulff, Schebesch 2006`, `Champneys,Sandstede 2007`)

Perspectives (continued)

- Multifronts and multipulses traveling at different speeds in 1D can be frozen independently (`WJB,Selle,Thümmler 2008`). Stability of the 'decompose and freeze' approach for multifronts and multipulses (`Selle 2009`),
- Generalization of the 'decompose and freeze' approach to dimensions ≥ 2.

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