Rock, Rattle and Slide

*bifurcation theory for piecewise-smooth systems*

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6: Conclusion
smooth bifurcation theory

\[
\dot{x} = f(x, \mu), \quad x \in D \subset \mathbb{R}^n, \quad \mu \in \mathbb{R}^p, \quad f \text{ smooth}
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Generates semiflow \( \Phi_{\mu}(x, t) \) and phase portrait = set of all trajectories \( \{\Phi(x, \cdot), \forall x \in D\} \).
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- **Analytic** Branch of invariant sets \( \Gamma(\mu) \). **Bifurcation** is a \( \mu \)-value where Implicit Function Theorem (IFT) fails.
  \( \Rightarrow \) **Branching** (Lyapunov-Schmidt reduction)
### smooth bifurcation theory

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  \( \Rightarrow \) **local bifurcations** Hopf, fold, flip, torus, . . .
  \( \Rightarrow \) **global bifurcations** homoclinic, tangency, crisis . . .

Classification by co-dimension
smooth bifurcation theory

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Classification by co-dimension

IFT & struct. stability need continuity & smoothness . . .
three types of nonsmoothness

- Impacting systems:
three types of nonsmoothness

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- Filippov systems:
three types of nonsmoothness

- Impacting systems:

- Piecewise smooth continuous systems:

- Filippov systems:
a motivating example

Oscillations of a pressure relief valve. Licsko, C. & Hös

noise at $\sim 375\, \text{Hz}$ at a range of flow speeds
a simple (dimensionless) model

\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\kappa y_2 - (y_1 + \delta) + y_3 \\
\dot{y}_3 &= \beta (q - \sqrt{y_3 y_1})
\end{align*}

\( y_1 > 0 \) valve displacement; \( y_2 \) valve velocity, \( y_3 \) pressure

\( \beta \) valve spring stiffness; \( \delta \) valve pre-stress
\( q \), flow rate; \( \kappa \), fluid damping

at \( y_1 = 0 \) apply a Newtonian restitution law:

\[ y_2(t_*) = -r y_2(t_*) \]

Low \( \kappa \) \( \Rightarrow \) limit cycles between 2 Hopf bifs \( q = q_{\min}, q_{\max} \).
brute force numerics

\[ \kappa = 1.25, \beta = 20, \delta = 10 \text{ (representative of experiment)} \]

Chaotic rattling due to Grazing events at \( q \approx 7.54, 5.95 \)
more realistic PDE model

Bazso, C. & Hös

Similar results including chattering at low pressure
A piecewise smooth (PWS) system is set of ODEs

\[ \dot{x} = F_i(x, \mu), \quad \text{if} \quad x \in S_i, \]
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Discontinuity set \( \Sigma_{ij} := S_i \cap S_j \) is \( \mathbb{R}^{(n-1)} \)-dim manifold \( \subset \partial S_j \cup \partial S_i \). Each \( F_i \) smooth in \( S_i \) generates flow \( \Phi_i(x, t) \).
Formalisms for nonsmooth system

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Degree of smoothness of \( x \in \Sigma_{ij} \) is order of 1st non-zero term in Taylor expansion of \( \Phi_i(x, t) - \Phi_j(x, t) \)
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impacting systems: deg. 0 need reset map

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PWS continuous systems: deg. \( \geq 2 \)

i.e. \( F_i(x) = F_j(x) \) but \( \exists k \geq 1 \) s.t. \( \frac{d^k F_i}{dx^k} \neq \frac{d^k F_j}{dx^k} \)
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- **Filippov systems** deg. 1. Have possibility of *sliding motion*. E.g. if \( \Sigma_{ij} := \{ H(x) = 0 \} \),
  \[(H_x F_1) \cdot (H_x F_2) < 0.\]
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- All smooth bifurcations can occur in PWS systems (because Poincaré map is typically analytic!)
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- Also discontinuity induced bifurcations (DIB) where invariant sets have non-structurally stable interaction with a $\Sigma_{ij}$. 
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- Can lead to classical (topological) bifurcation or not
bifurcation

- All smooth bifurcations can occur in PWS systems (because Poincaré map is typically analytic!)
- Also discontinuity induced bifurcations (DIB) where invariant sets have non-structurally stable interaction with a $\Sigma_{ij}$.
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idea topological DIB $\iff$ PW structural stability
types of DIB

Boundary equilibrium bifurcations
types of DIB

- Boundary equilibrium bifurcations
- Grazing bifurcations of limit cycles
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this talk: periodic orbit DIBs

Goal: Catalogue & unfold codim-1 possibilities. E.g. ‘grazing bifurcation’: [Nordmark]
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Derive map close to DIB as composition of smooth Poincaré map $P_\pi$ and discontinuity mapping PDM

$$\Sigma : \{H(x) = 0\}$$
this talk: periodic orbit DIBs

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E.g. ‘grazing bifurcation’: [Nordmark]

Derive map close to DIB as composition of smooth Poincaré map $P_\pi$ and discontinuity mapping $PDM$

Use results on border collisions of maps to classify dynamics [Feigin] [Yorke, Banergee et al]
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\[ \Sigma : \{ H(x) = 0 \} \]

- Use results on border collisions of maps to classify dynamics [Feigin] [Yorke, Banerjee et al]

- Nb. piecewise linear (PWL) flow $\not\Rightarrow$ PWL map
2. Grazing bifurcation in impact systems


- Consider single impact surface \( \Sigma := \{ H(x) = 0 \} \) with impact law:

\[
x^+ = R(x^-) = x^- + W(x^-)H_x F(x^-)
\]

\( W \) is smooth function and \( H_x F(x^-) \) is ‘velocity’. e.g.

\[
W = -(1 + r)H_x \Rightarrow \text{Newton’s ‘restitution law’}
\]

- More complex impact laws are possible, e.g. impact with friction (see later)
discontinuity mapping (PDM)

- **PDM**: $x_1 \mapsto x_5$ maps Poincaré section
  \[ \Pi = \{ H_x F(x) = 0 \} \]
  to itself

- Computes correction to trajectory as if $\Sigma$ were absent
explicit form of PDM

cf. [Fredrickson & Nordmark]

\[ x \mapsto \begin{cases} 
  x & \text{if } H(x) \geq 0 \\
  x + \beta(x, y)y & \text{if } H(x) < 0
\end{cases} \]

where \[ \beta = -\sqrt{2a} \left( W - \frac{(H_x F)_x W}{a} F \right) + O(y^2) , \]

where \( y = \sqrt{-H} \) and

\[ a(x) = d^2 H / dt^2 = (H_x F)_x F = H_{xx} F F + H_x F_x F \]

\( \Rightarrow \) square root map
Proof is by Taylor expansion of flow in \((x, y)\) and IFT
Proof is by Taylor expansion of flow in $(x, y)$ and IFT

Use PDM to correct non-grazing Poincaré map $P_{\pi}$:

$$P_N = P_{\pi} \circ P_{PDM}$$

$$P_N(x, \mu) = M_1 x + N \mu + O(x^2, \mu^2) \quad \text{if} \quad H(x) > 0$$

$$= M_2 x + N \mu + B \sqrt{|H(x)|} + O(x^2, \mu^2) \quad H(x) < 0$$
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Conditions on \(M_{1,2}, B, C\) for given periodic orbit to exist
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- Attractor for \(\mu > 0\) depends on linearisation of orbit for \(\mu < 0\).
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Attractor for \(\mu > 0\) depends on linearisation of orbit for \(\mu < 0\).

Simplest case:
\(\lambda_1\) real leading eigenvalue of \(M_1\) . . ., then dynamics is determined by 1D map:
dynamics of 1D map

\[ f(x) = \sqrt{\mu - x} + \lambda_1 \mu \quad x < \mu, \quad f(x) = \lambda_1 x \quad x > \mu, \]

1. \( \frac{2}{3} < |\lambda_1| < 1 \): robust chaotic attractor size \( \sim \sqrt{\mu} \).

2. If \( \frac{1}{4} < |\lambda_1| < \frac{2}{3} \) alternating series of chaos and period-\( n \) orbits, \( n \to \infty \) as \( \mu \to 0 \).

3. \( 0 < |\lambda_1| < \frac{1}{4} \): just period-adding cascade
return to Ex.i: valve rattle

Two grazing bifurcation events $q = 7.54$, $q = 5.95$

$q = 5.95$: $\lambda_1 < 0 \Rightarrow$ discontinuous jump in attractor
$q = 7.54$: $\lambda_1 = 0.8537 \Rightarrow$ jump to chaos. Iterate map
3. DIBs in PWS continuous systems

Simplest case: grazing bifurcation
Analyse using discontinuity mapping: PDM
# general results for PDM

<table>
<thead>
<tr>
<th>degree</th>
<th>$F$</th>
<th>jump</th>
<th>Uniform Case</th>
<th>Non-uniform</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\delta$-function</td>
<td>$x$</td>
<td>square-root</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>bounded</td>
<td>$F$</td>
<td></td>
<td>$O(1/2)$</td>
</tr>
<tr>
<td>2</td>
<td>$C^0$</td>
<td>$F_x$</td>
<td>$O(3/2)$</td>
<td>$O(3/2)$-type</td>
</tr>
<tr>
<td>3</td>
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<td>$F_{xx}$</td>
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</tbody>
</table>

$F(x)$ continuous at $\Sigma \Rightarrow$ no immediate jump in attractor
1D map with $O(3/2)$ singularity

- Consider [Halse, di Bernardo et al]

$$x \mapsto \begin{cases} 
\nu x - \mu & x \leq 0 \\
\nu x + \eta x^{3/2} - \mu & x > 0 
\end{cases}$$

- $0 < \nu < 1 \Rightarrow$ simple fixed point. No bifurcation at $\mu = 0$.

- but with $\eta < 0$ get nearby fold at $\mu = -\frac{4(1-\nu)^3}{3\eta^2}$ (much closer than smooth fold if $\nu \approx 1$)
Also, get **period-adding cascades**. E.g. for $\gamma = 3/2$, $\eta = -1$. Then stable $L^{k-1}R$ orbits exist for

$$\frac{-8(\nu^k + 1)^3 - 12(1 - \nu^k)(1 + \nu^k)^2}{27\nu^2(k-1)(1 + \nu + \nu^2 + \ldots + \nu^{k-1})} < \mu < -\left(\frac{\nu^k - 1}{\nu^{k-1}(\nu - 1)}\right)^2.$$  

**case $\gamma = 2$:**
Ex. ii: A realistic stick-slip oscillator

Dankowicz 1999

\[ y_1 \text{ - horizontal displacement; } y_2 = \dot{y}_1 \]
\[ y_3 \text{ - vertical displacement; } y_4 = \dot{y}_3 \]
\[ y_5 \text{ - shear deformation of asperities} \]
\[ \text{belt velocity } U = 1 \]
equations of motion

\[ \begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -1 + \left[ 1 - \gamma U |1 - y_4| y_2 + \beta U^2 (1 - y_4)^2 \sqrt{K(y_1)} \right] e^{y_1 - d}, \\
\dot{y}_3 &= y_4, \\
\dot{y}_4 &= -s y_3 + \frac{\sqrt{g \sigma}}{U} e^{-d} \left[ \mu (y_5 e^{-y_1} - 1) + \alpha U^2 S(y_1, y_4) \right], \\
\dot{y}_5 &= \frac{1}{\tau} \left[ (1 - y_4) - |1 - y_4| y_5 \right],
\end{align*} \]

where \( K(y_1) = 1 - \frac{y_1 - d}{\Delta} \),
\( S(y_1, y_4) = (1 - y_4) |1 - y_4| K(y_1) e^{-y_1} - 1 + \frac{d}{\Delta} \).

\( \Rightarrow \) PWS continuous across discontinuity boundary \( y_4 = 1 \).
grazing bifurcation analysis

Dankowicz & Nordmark 2000
3 successive zooms of bifurcation diagram:
grazing bifurcation analysis

Dankowicz & Nordmark 2000
Simulation (left) and iteration of DM (right)

in local map co-ordinates $\sim y_4 \times 10^{-4}$
4. Sliding DIBs in Filippov systems

Kowalczyk, Nordmark, diBernardo
Four possible DIB involving collision of limit cycle with sliding boundary $\partial \hat{\Sigma}^-$; see Mike Jeffrey’s talk

(a) crossing sliding
(b) grazing sliding
(c) switching sliding
(d) adding sliding
Unfold with discontinuity mapping Di Bernardo, Kowalczyk, Nordmark

<table>
<thead>
<tr>
<th>Bifurcation type</th>
<th>DM leading-order term</th>
<th>Map singularity</th>
</tr>
</thead>
<tbody>
<tr>
<td>crossing sliding</td>
<td>$\varepsilon^2 + O(\varepsilon^3)$</td>
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</tr>
<tr>
<td>grazing sliding</td>
<td>$\varepsilon + O(\varepsilon^{3/2})$</td>
<td>1</td>
</tr>
<tr>
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<td>$\varepsilon^3 + O(\varepsilon^4)$</td>
<td>3</td>
</tr>
<tr>
<td>adding sliding</td>
<td>$\varepsilon^2 + O(\varepsilon^{5/2})$</td>
<td>2</td>
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</table>

- Maps are non-invertible on one side
- Only grazing sliding $\Rightarrow$ jump in attractor
Ex.iv: a relay control system

\[ \dot{x} = Ax - B\text{sgn}(y), \quad y = C^T x, \]

\[
A = \begin{pmatrix}
-a_1 & 1 & 0 \\
-a_2 & 0 & 1 \\
-a_3 & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}, \quad C^T = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}^T.
\]

Complex dynamics:

\[ b = (1, -2, 1)^T, \quad a_{31} = -5, \quad a_{21} = -99.3, \quad \text{and} \]

(a) \( a_{11} = 1.206, 1.35, \) periodic; (b) nearby, chaotic
bifurcation diagram

A grazing-sliding cascade
5. Impact with friction

Dankowicz Nordmark & C.

- $q \in \mathbb{R}^n$, with rigid contact in 2D + Coulomb friction

$$M(q, t) \ddot{q} = f(q, \dot{q}, t) + \lambda_T c_u^T(q, t) + \lambda_N c_v^T(q, t),$$

- Scalar constraint $y \geq 0$, $y \in \mathbb{R}$ normal distance;
  $\lambda_N \geq 0$, $\lambda_T \in \mathbb{R}$ normal and tangential forces;

- Coulomb friction, $|\lambda_T| \leq \mu \lambda_N$, $\lambda_T = -\text{sign}(u)\mu \lambda_N$ if $u \neq 0$

- e.g. rod & table Painlevé 1905, Brogliato et al.
contact dynamics

- Project Lagrangian onto $u$ and $v$ directions:

$$\dot{u} = a(q, \dot{q}, t) + \lambda_T A(q, t) + \lambda_N B(q, t),$$

$$\dot{v} = b(q, \dot{q}, t) + \lambda_T B(q, t) + \lambda_N C(q, t),$$

$$A = c_u \cdot M^{-1} \cdot c_u^T, \quad B = c_u \cdot M^{-1} \cdot c_v^T, \quad C = c_v \cdot M^{-1} \cdot c_v^T,$$

- positive definite $M \Rightarrow A > 0 \quad C > 0, \quad AC - B^2 > 0$

- special case $B = 0 \Rightarrow \text{“independent” normal and tangential motion} \Rightarrow \text{can use Newtonian restitution}$

$$v \rightarrow -rv \text{ at impact (well posed)}$$

- what if $B \neq 0$?, e.g. for rod example ($l = 2, m = 2$):

$$A = 1 + 3 \sin^2 \theta, \quad B = 3 \sin 2\theta, \quad C = 1 + 3 \cos^2 \theta$$
modes of sustained motion

**free flight:** $y > 0$. No contact forces:

$$(\lambda_T, \lambda_N) = (0, 0).$$

**positive/negative slip:** $y = 0$, $v = 0$, $\lambda_N > 0$, $u \neq 0$. Full friction \( \lambda_T = -\text{sign}(u) \mu \lambda_N. \)

$$(\lambda_T, \lambda_N) = \frac{b}{C - \text{sign}(u) \mu B} (\text{sign}(u) \mu, -1).$$

**stick:** $y = 0$, $v = 0$, $\lambda_N > 0$, $u = 0$, $|\lambda_T| < \mu \lambda_N.$

$$(\lambda_T, \lambda_N) = \frac{1}{AC - B^2} (bB - aC, aB - bA).$$
impacts

- **Def:** impact phase infinitesimal time intervals in which $\lambda_N$ and $\lambda_T$ are impulses (distributions)

- **key idea:** re-scale $\tau = t/\varepsilon$, $\Lambda_{N,T} = \varepsilon\lambda_{N,T} = O(1)$ and let $\varepsilon \rightarrow 0$.

- impact-phase dynamics: $q' = 0$ and

\[
\begin{align*}
u' &= A\Lambda_T + B\Lambda_N, \\
v' &= B\Lambda_T + C\Lambda_N
\end{align*}
\]

$(A, B, C$ are constant during impact since $q' = 0$.

- integrating $I_{N,T} = \int_\text{impact} \Lambda_{N,T} d\tau$ gives:

\[
(I_T, I_N) = \frac{1}{AC - B^2} (C\Delta u - B\Delta v, A\Delta v - B\Delta u).
\]

Change in $\dot{q}$ is then: $\Delta \dot{q} = M^{-1}(c_u^T I_T + c_v^T I_N)$
but how to compute $\Delta u, \Delta v$?

\[ u' = A\Lambda_T + B\Lambda_N, \quad v' = B\Lambda_T + C\Lambda_N, \]

$\Rightarrow$ 3 modes of impulsive motion:

**impulsive positive slip:** $u > 0$. Full friction $\lambda_T = -\mu\lambda_N$.

**impulsive negative slip:** $u < 0$. Full friction $\lambda_T = \mu\lambda_N$.

**impulsive stick:** $u = 0$, $|\lambda_T| < \mu\lambda_N$. Only possible if $|B| < \mu A$.

$\Rightarrow$ For all modes: $u' = k_u \lambda_N$, $v' = k_v \lambda_N$ where

\[
(k_u, k_v) = (k_u^+, k_v^+) = (B - \mu A, C - \mu B) \quad \text{for pos. slip}
\]

\[
(k_u, k_v) = (k_u^-, k_v^-) = (B + \mu A, C + \mu B) \quad \text{for neg. slip}
\]

\[
(k_u, k_v) = (k_u^0, k_v^0) = (0, \frac{AC - B^2}{A}) \quad \text{for stick}
\]
when is the impact finished?

3 possibilities:

1. **Newtonian coefficient of restitution**
   Relate post-impact velocities to pre-impact:
   \[ v_1 = -rv_0 \]

2. **Poisson coefficient of restitution (Glocker)**
   Relate normal impulses during compression and restitution:
   \[ I_r = -rI_c \]

3. **Energetic coefficient of restitution (Stronge)**
   Relate normal-force work during compression and restitution:
   \[ W_r = -r^2W_c \]

If impact phase has a single mode \( \Rightarrow \) all 3 agree.
But (Stewart) 1 & 2 may **increase** kinetic energy for \( r < 1 \).
Hence we use 3 & derive explicit formulae (cf. Stronge)
impulsive motion follows straight lines

\[ k_{u}^{+} < 0 \]
\[ k_{v}^{+} > 0 \]
\[ k_{u}^{-} > 0 \]
\[ k_{v}^{-} > 0 \]
discontinuity-induced bifurcation

- dynamics cross region boundary as parameters vary
- \( \Rightarrow \) hybrid flow map can be \( C^1 \) (no bifurcation) or \( C^0 \) (jump in multipliers)
- e.g. loss of period-one impacting periodic orbit

rod example with Van-der-pol type forcing:

\[
S_x = -k_1(x - u_{dr}t) - c_1(u - u_{dr}) \\
S_y = -k_2(y - y_0) - c_2(y - y_0)^2 - y_1^2)R = -k_3(\theta - \theta_0) - c_3\theta
\]
ambiguities during sustained motion

To to simulate as a hybrid system, need to resolve:

A. Painlevé paradox for slip  If \( y = 0, v = 0, b > 0 \) and \( C - \mu B < 0, u > 0 \) (or \( C + \mu B < 0, u < 0 \)), then motion could continue with
- Sustained free flight
- Sustained positive (negative) slip
- An impact with zero initial normal velocity

B. Painlevé paradox for stick  If \( y = 0, v = 0, u = 0, b > 0, \)
\[ |bB - aC| < \mu(aB - bA) \] and \( C - \mu B < 0 \) (or \( C + \mu B < 0 \)), then motion could continue with
- Sustained free flight
- Sustained stick
show consistency via smoothing

- Introduce constitutive relation $\lambda_N(y, v)$ that is “stiff”, “restoring”, and “dissipative”.

- Case A slip (WLOG positive slip),

\[
\begin{align*}
\dot{y} &= v, \\
\dot{v} &= b + (C - \mu B)\lambda_N(y, v).
\end{align*}
\]

$b > 0$, $C - \mu B < 0 \Rightarrow$ large negative stiffness, 
$\Rightarrow$ slipping will never occur, must immediately lift off ($y > 0$) or take impact ($y < 0$)

- Case B stick $\dot{v} = \frac{(bA - aB) + (AC - B^2)\lambda_N(y, v)}{A}$

$\Rightarrow$ always large positive “stiffness” hence vertical motion is asymptotically stable ($even if b > 0$)
ambiguities at mode transitions

Sustained motion is consistent BUT what about transitions

- **Case a.** approach to the Painlevé boundary \( (C - \mu B = 0) \) during (positive) slip.

  previous analysis shows: can’t actually reach \( C - \mu B = 0 \), so what happens instead?

- **Case b.** transitions into stick or chatter

  **Def:** chattering (also known as zeno-ness) is accumulation of impacts. No contradiction if accumulate in forwards time. But can get reverse chatter.
a. unfolding $C - \mu B \rightarrow 0$ while slipping

cf. Genôt & Brogliato

- **Re-scale time** $t = (C - \mu B)s \Rightarrow$

$$\frac{d}{ds} \begin{pmatrix} C - \mu B \\ b \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} C - \mu B \\ b \end{pmatrix}$$

- **Eigenvector** $(0, 1)^T \Rightarrow$ trajectory tend to $C - \mu B = 0$, only if $b = 0$
approaching the singular point
what happens after singular point?

- could lift off, or take a (zero-velocity) impact.
- e.g. simulate example for stiff, compliant contact force

\[ \lambda_N(y, v) = \frac{(1 + r^2) - (1 - r^2) \tanh \left( \frac{v}{\delta} \right)}{2} \left( -\frac{y}{\varepsilon} \right) \]

for small \( \delta, \varepsilon \)

- resolvable (ongoing work) \( \Rightarrow (?) \) impact always occurs
b. transition into stick or chatter

e.g. nearby initial conditions with \( b < 0 \)

\[ v \rightarrow ev \] after impact + lift off.
analysis of chatter

Find parameter regions in which $e > 1$ (reverse chatter) despite $r < 1$ - even in the “non-Painlevé” case
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  - catastrophic sliding bifurcations ∼ canards Jeffrey, C. di Bernardo, Shaw, Moehlis
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